

QUANTUM COHOMOLOGY OF A PFAFFIAN CALABI-YAU VARIETY: VERIFYING MIRROR SYMMETRY PREDICTIONS

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To my parents on their 65th and 70th birthday.

ABSTRACT. We formulate a generalization of Givental-Kim's quantum hyperplane principle. This is applied to compute the quantum cohomology of a Calabi-Yau 3-fold defined as the rank 4 locus of a general skew-symmetric 7×7 matrix with coefficients in \mathbf{P}^6 . The computation verifies the mirror symmetry predictions of Rødland [25].

0. INTRODUCTION

The rank 4 degeneracy locus of a general skew-symmetric 7×7 -matrix with $\Gamma(\mathcal{O}_{\mathbf{P}^6}(1))$ -coefficients defines a non-complete intersection Calabi-Yau 3-fold M^3 with $h^{1,1} = 1$. We recall some results of Rødland [25] on the mirror symmetry of M^3 : a potential mirror family W_q is constructed as (a resolution of) the orbifold M_q^3/\mathbf{Z}_7 , where M_q^3 is a one-parameter family of invariants of a natural \mathbf{Z}_7 -action on the space of all skew-symmetric 7×7 -matrices. It is shown that the Hodge-diamond of W_q mirrors the one of M^3 . Further, at a point of maximal unipotent monodromy¹, the Picard-Fuchs operator for the periods is computed to be (with $D = q \frac{d}{dq}$):

$$(1) \quad \begin{aligned} & (1 - 289q - 57q^2 + q^3)(1 - 3q)^2 D^4 \\ & + 4q(3q - 1)(143 + 57q - 87q^2 + 3q^3) D^3 \\ & + 2q(-212 - 473q + 725q^2 - 435q^3 + 27q^4) D^2 \\ & + 2q(-69 - 481q + 159q^2 - 171q^3 + 18q^4) D \\ & + q(-17 - 202q - 8q^2 - 54q^3 + 9q^4). \end{aligned}$$

Mirror symmetry conjectures that this operator is equivalent to the operator

$$(2) \quad D^2 \frac{1}{K} D^2 \quad \text{where} \quad K(q) = 14 + \sum_{d \geq 1} n_d d^3 \frac{q^d}{1 - q^d},$$

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¹There are two points with maximal unipotent monodromy. Remarkably, the Picard-Fuchs equation at the other point is the one found in [2] for the mirror of the complete intersection Calabi-Yau 3-fold in $G(2, 7)$.

and n_d , the *instanton number of degree d rational curves on M^3* , is defined [20] using Gromov-Witten invariants by

$$\langle p, p, p \rangle_d^{M^3} = \sum_{k|d} k^3 n_k.$$

We shall prove the conjecture.

Theorem 1. *The differential operators (1) and (2) are equivalent under mirror transformations. That is:*

Let I_0, I_1, I_2, I_3 be a basis of solutions to (1) with holomorphic solution $I_0 = 1 + \sum_{d \geq 1} a_d q^d$ and logarithmic solution $I_1 = \ln(q)I_0 + \sum_{d \geq 1} b_d q^d$. Then

$$\frac{I_0}{I_0}, \frac{I_1}{I_0}, \frac{I_2}{I_0}, \frac{I_3}{I_0},$$

is a basis of solutions for (2) after change of coordinates $q = \exp(\frac{I_1}{I_0})$.

Our approach follows closely the work of Givental [15, 14] for complete intersections in toric manifolds, and Batyrev, Ciocan-Fontanine, Kim, van Straten [2, 1] for complete intersections in partial flag manifolds. It builds on the following three observations:

- i) A well-known construction identifies the degeneracy locus M^3 with the vanishing locus of a section of a vector bundle on a Grassmannian manifold (see Section 2). It is crucial, for us, that this vector bundle decomposes into a direct sum of vector bundles $E \oplus H$, where H is again a direct sum of line bundles.
- ii) The quantum hyperplane principle of B. Kim [18] extends to relate the E -restricted quantum cohomology with the $E \oplus H$ -restricted one. This is formulated as a general principle in Section 1.
- iii) The E -restricted quantum cohomology can be effectively computed using localization techniques and WDVV-relations. An application of the quantum hyperplane principle then yields Theorem 1. The computations are carried out in Section 2.

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1. GROMOV-WITTEN THEORY

We begin by recalling some basic results on $g = 0$ Gromov-Witten invariants before stating the quantum hyperplane principle. Our approach is the algebraic one following [19]. We refer the reader to [11, 7] for a fuller account and references.

Frobenius rings. Let X be a smooth projective variety over \mathbf{C} . Unless otherwise specified, we only consider even dimensional cohomology with

²<http://www.math.uio.no/~einara/Maple/Maple.html>.

rational coefficients. In fact we will work with a further restriction: if E is a vector bundle on X , and Y is the zero-set of a regular section of E , then we are mainly interested in the cohomology classes on Y that are pulled back from X . These are represented by the graded Frobenius ring $A^*(E)$ with

$$(3) \quad A^p(E) := H^{2p}(X, \mathbf{Q}) / \text{ann}(E_0)$$

and non-degenerate pairing $\langle \gamma_1, \gamma_2 \rangle^E := \int_X \tilde{\gamma}_1 \tilde{\gamma}_2 E_0$, where E_0 is the top Chern class of E , and $\tilde{\gamma}_i$ denotes a lift of γ_i to $A^*(X)$.

Let $A_1(E, \mathbf{Z})$ be the dual of $A^1(E, \mathbf{Z})/\text{torsion}$. We will identify $A_1(E, \mathbf{Z})$ with the image of the natural inclusion

$$(4) \quad A_1(E, \mathbf{Z}) \rightarrow A_1(X, \mathbf{Z}).$$

Moduli space of stable maps [20, 11]. Let (C, s_1, \dots, s_n) be an algebraic curve of arithmetic genus 0 with at worst nodal singularities and n nonsingular marked points. A map $f: C \rightarrow X$ is stable if all contracted components are stable (i.e. each irreducible component contains at least three *special* points, where special means marked or singular). For $d \in A_1(X, \mathbf{Z})$, let $X_{n,d}$ denote the coarse moduli space (or Deligne-Mumford stack) of stable maps with $f_*[C] = d$. If X is *convex*, that is $H^1(C, f^*TX) = 0$ for all stable maps, then $X_{n,d}$ is an orbifold (only quotient singularities) of complex dimension

$$(5) \quad \dim_{\mathbf{C}} X + \int_d c_1(X) + n - 3.$$

Of great importance to the theory are some natural maps on the moduli space of stable maps. For $i = 1, \dots, n$, let $e_i: X_{n,d} \rightarrow X$ be the map obtained by evaluating stable maps at s_i , and let $\pi_i: X_{n,d} \rightarrow X_{n-1,d}$ be the map which forgets the marked point s_i . Also of significance are certain gluing maps which stratify the boundaries of the moduli spaces. In the stack theoretic framework, the diagram

$$(6) \quad \begin{array}{ccc} X_{n+1,d} & \xrightarrow{e_{n+1}} & X \\ \pi_{n+1} \downarrow & & \\ X_{n,d} & & \end{array}$$

along with sections $s_i: X_{n,d} \rightarrow X_{n+1,d}$ defined by requiring $e_i = e_{n+1} \circ s_i$, is identical to the *universal* stable map.

Gromov-Witten invariants [27, 24, 19, 21, 5, 4]. Suppose E is a convex vector bundle (i.e. $H^1(C, f^*E) = 0$ for all stable maps). Base change theorems in [16] imply that $\pi_{n+1,*} e_{n+1}^* E$ is a vector bundle on $X_{n,d}$ with fibers $H^0(C, f^*E)$. Let $E_{n,d}$ denote the top Chern class of $\pi_{n+1,*} e_{n+1}^* E$ and let c_i denote the first Chern class of the line bundle $s_i^* \omega_{\pi_{n+1}}$ on $X_{n,d}$, where $\omega_{\pi_{n+1}}$ is the relative sheaf of differentials. Let c be an indeterminate.

A system of *E-restricted Gromov-Witten invariants* for X is the family of multilinear functions $\langle \cdot \rangle_d^E$ on $A^*(E)[[c]]^{\otimes n}$, defined for all $n \geq 0$ and $d \in A_1(E, \mathbf{Z})$ by

$$(7) \quad \langle P_1 \gamma_1, \dots, P_n \gamma_n \rangle_d^E := \int_{[X_{n,d}]} \prod_{i=1}^n P_i(c_i) e_i^*(\tilde{\gamma}_i) E_{n,d},$$

where $\gamma_i \in A^*(E)$, $P_i \in \mathbf{Q}[[c]]$, and $[X_{n,d}]$ is the *virtual fundamental class* of dimension (5).

There is an exact sequence of vector bundles

$$(8) \quad 0 \rightarrow \ker \rightarrow \pi_{n+1,*} e_{n+1}^* E \rightarrow e_i^* E \rightarrow 0,$$

where the right-hand map is obtained by evaluating sections at the i -th marked point. This implies that $E_{n,d}$ is divisible by E_0 in $A^*(X_{n,d})$, hence the invariants (7) are independent of the chosen lifts $\tilde{\gamma}_i$.

When X is convex, then $[X_{n,d}]$ is simply the fundamental class of $X_{n,d}$. If $Y \subset X$ is cut out by a regular section of E , then

$$(9) \quad j_* \sum_{i_* d' = d} [Y_{n,d'}] = E_{n,d} \cdot [X_{n,d}],$$

where the map $j: Y_{n,d'} \rightarrow X_{n,d}$ is induced from the inclusion map $i: Y \rightarrow X$.

Let $\{\Delta_i\}$, $\{\Delta^i\}$ denote a pair of homogeneous bases of $A^*(E)$ such that $\langle \Delta_i, \Delta^j \rangle^E = \delta_i^j$, and let $T_i \in A^*(E)[[c]]$. The natural maps on the moduli space of stable maps respect the virtual classes, hence induce important relations on GW-invariants. Among these are:

Divisor equation. For $p \in A^1(E)$ we have

$$\begin{aligned} \langle p, T_1, \dots, T_n \rangle_d^E &= \\ \left(\int_d \tilde{p} \right) \langle T_1, \dots, T_n \rangle_d^E &+ \sum_{i=1}^n \langle T_1, \dots, pT_i/c, \dots, T_n \rangle_d^E. \end{aligned}$$

WDVV-relation. Denote³

$$\left(\begin{smallmatrix} T_1 & T_4 \\ T_2 & T_3 \end{smallmatrix} \right)_d := \sum_{d_1+d_2=d} \langle T_1, T_2, \Delta_i \rangle_{d_1}^E \langle \Delta^i, T_3, T_4 \rangle_{d_2}^E.$$

Then,

$$\left(\begin{smallmatrix} T_1 & T_4 \\ T_2 & T_3 \end{smallmatrix} \right)_d = \left(\begin{smallmatrix} T_1 & T_4 \\ T_3 & T_2 \end{smallmatrix} \right)_d.$$

Topological recursion relation (TRR).

$$\langle T_1, T_2, T_3 \rangle_d^E = \sum_{d_1+d_2=d} \langle T_1/c, \Delta_i \rangle_{d_1}^E \langle \Delta^i, T_2, T_3 \rangle_{d_2}^E.$$

The above equations are subject to some restrictions: If $d = 0$ we must assume that $n \geq 3$ in the divisor equation. Further, all undefined correlators appearing above are set to 0, except for $\gamma_1, \gamma_2 \in A^*(E)$ we define

$$\langle \gamma_1/c, \gamma_2 \rangle_0^E := \langle \gamma_1, \gamma_2 \rangle^E.$$

Let $\{p_i\}$ be a *nef* (i.e. pairs non-negatively with all effective curve classes in $A_1(E, \mathbf{Z})$) basis for $A^1(E, \mathbf{Z})/\text{torsion}$, and let $\{q_i\}$ be formal homogeneous parameters such that $\sum_i \deg(q_i) p_i = c_1(X) - c_1(E)$ modulo $\text{ann}(E_0)$. The WDVV-relations imply the associativity of the *quantum product* defined by

$$\Delta_i *_E \Delta_j := \sum_{d,k} \langle \Delta_i, \Delta_j, \Delta_k \rangle_d^E q^d \Delta^k,$$

³We use the Einstein summation convention.

where $q^d = \prod_i q_i^{\int_d \tilde{p}_i}$. Note that the product is homogeneous with the chosen grading.

Quantum hyperplane principle. Let \hbar be a formal homogeneous variable of degree 1. Let e_{1*}^E be the map induced from the push-forward e_{1*} by passing to the quotient (3). Use $E'_{1,d}$ to denote the top Chern class of the kernel in (8), thus $E_{1,d} = E_0 E'_{1,d}$. Consider the following degree 0 vector in $A^*(E)[[q, \hbar^{-1}]]$:

$$\begin{aligned} J_E &:= e^{p \ln(q)/\hbar} \sum_d q^d e_{1*}^E \left(\frac{E'_{1,d}}{\hbar(\hbar - c_1)} \right) \\ (10) \quad &= e^{p \ln(q)/\hbar} \sum_d q^d < \frac{\Delta_i}{\hbar(\hbar - c)} >_d^E \Delta^i, \end{aligned}$$

where $p \ln q = \sum_i \ln(q_i) p_i$, and the convention $< \frac{\Delta_i}{\hbar(\hbar - c)} >_0^E \Delta^i = 1$ is used. Suppose $H = \oplus L_i$ is a sum of convex line bundles on X . If $c_1(X) - c_1(E \oplus H)$ is nef, the quantum hyperplane principle suggests an explicit relationship between J_E and $J_{E \oplus H}$ via the following adjunct in $A^*(E \oplus H)[[q, \hbar^{-1}]]$:

$$(11) \quad I_E^H := e^{p \ln(q)/\hbar} \sum_d q^d H_d e_{1*}^{E \oplus H} \left(\frac{E'_{1,d}}{\hbar(\hbar - c_1)} \right)$$

where

$$H_d := \prod_i \prod_{m=1}^{\int_d c_1(L_i)} (c_1(L_i) + m\hbar),$$

and the q_i 's are regraded so that $\sum_i \deg(q_i) p_i = c_1(X) - c_1(E \oplus H)$ modulo $\text{ann}(E_0)$. Hence, by assumption, all $\deg(q_i) \geq 0$. The precise statement is:

The vectors I_E^H and $J_{E \oplus H}$ coincide up to a mirror transformation of the following type:

- i) *multiplication by $a \exp(b/\hbar)$ where a and b are homogeneous q -series of degree 0 and 1 respectively,*
- ii) *coordinate changes*

$$\ln q_i \mapsto \ln q_i + f_i,$$

where f_i are homogeneous q -series of degree 0 without constant term.

Further, the mirror transformation is uniquely determined by the first two coefficients in the \hbar^{-1} -Taylor expansion of I_E^H and $J_{E \oplus H}$.

Theorem 2. *If X is a homogenous space and E is equivariant with respect to a maximal torus action on X , then the quantum hyperplane conjecture as formulated above is true.*

Proof. The proof in [18] for $\text{rank}(E) = 0$ extends with minor modifications to the general case. \square

Remark 1. An early version of this principle appeared in [3]. Givental formulated and proved the $\text{rank}(E) = 0$ case for toric manifolds [15, 14]. The above formulation when $\text{rank}(E) = 0$ is due to B. Kim [18]. Extending

the conjecture of B. Kim we expect that the principle holds for more general X . In [26] the conjecture was tested on a non-convex non-toric manifold.

Remark 2. An analogue generalization of the hyperplane principle in [13, 17] for *concave* H can be formulated with E convex/concave. The proof in [18] extends to cover these cases when X is homogenous. See also [22].

Differential equations. The vector J_E encodes all the E -restricted one-point GW-invariants. Reconstruction using TRR [23] shows that these are determined by two-point GW-invariants without c 's. This is organized nicely in terms of differential equations [8, 15]. Consider the *quantum differential equation*

$$(12) \quad \hbar q_k \frac{d}{dq_k} T = p_k *_E T, \quad k = 1, \dots, \text{rank}(A^1(E)),$$

where T is a series in the variables $\ln q_i$ and \hbar^{-1} with coefficients from $A^*(E)$. The WDVV-relations imply that the system is solvable. An application of the divisor equation and TRR [26, 23] shows that $J_E = \langle S, 1 \rangle^E$ for fundamental solution S of (12). In particular, if $c_1(X) - c_1(E)$ is *positive* the hyperplane principle, when true, yields an algebraic representation of the (*quantum*) \mathcal{D} -module generated by $J_{E \oplus H}$.

Remark 3. A useful application of Theorem 2 is the following: for partial flag manifolds $F = F(n_1, \dots, n_r, n)$ with universal sub-bundles U_{n_i} and quotient bundles Q_{n_i} , one may consider vector bundles E that are direct sums of bundles of type

$$\wedge^{p_1} U_{n_{i_1}}^\vee \otimes \wedge^{p_2} Q_{n_{i_2}} \otimes S^{p_3} U_{n_{i_3}}^\vee \otimes S^{p_4} Q_{n_{i_4}}.$$

If $c_1(F) - c_1(E)$ is positive, the E -restricted quantum cohomology can in principle be computed using localization techniques, and the theorem will yield the quantum D -module for the nef (and in particular the Calabi-Yau) cases of type $E \oplus H$, with H decomposable.

2. THE PFAFFIAN VARIETY

Let V be a vector space of dimension 7 and consider the projective space $P = \mathbf{P}(\wedge^2 V)$ with universal 7×7 skew-symmetric linear map

$$\alpha: V_P^\vee(-1) \rightarrow V_P,$$

where V_P denotes the trivial vector bundle on P with fiber V . Define $M \subset P$ as the locus where $\text{rank } \alpha \leq 4$. The scheme structure is determined by the Pfaffians of the diagonal 6×6 -minors. The variety is locally Gorenstein of codimension 3 in P with canonical sheaf $\mathcal{O}_M(-14)$. Its singular locus, which is the rank 2 degeneracy locus of α , is of codimension 7 in M [6]. This implies that the intersection $M^k = M \cap \mathbf{P}^{k+3}$ with a general linear sub-space \mathbf{P}^{k+3} in P is of dimension k , has canonical sheaf $\mathcal{O}_{M^k}(3-k)$, and is smooth when $k \leq 6$.

We recall a classic construction for degeneracy loci (see for instance [12], Example 14.4.11). Let $G = \text{Grass}_4(V)$ be the Grassmannian of 4-planes in V with universal exact sequence

$$0 \rightarrow U \rightarrow V_G \rightarrow Q \rightarrow 0.$$

Pulling everything back to $PG = P \times G$, we regard $\wedge^2 U(1)$ as a sub-bundle of $\wedge^2 V_{PG}(1)$, where the twists are with respect to $\mathcal{O}_P(1)$. The map α induces a regular section $\bar{\alpha}$ of the convex rank 15 quotient bundle on PG

$$A := \wedge^2 V_{PG}(1) / \wedge^2 U(1).$$

Lemma 1. *The zero-scheme $V(\bar{\alpha}) \subset PG$ projects birationally onto M . Moreover, the projection is isomorphic over the non-singular locus of M .*

For a general linear sub-space \mathbf{P}^{k+3} in P , let $(A^k, \bar{\alpha}_k)$ denote the pull-back of the pair $(A, \bar{\alpha})$ to $\mathbf{P}^{k+3} \times G$. By Lemma 1, $V(\bar{\alpha}_k)$ projects isomorphically to M^k for $k \leq 6$.

We are now set to compute the quantum \mathcal{D} -module of the Calabi-Yau variety M^3 using Theorem 2. This can in principle be done from any of the A^k -restricted ($k \geq 4$) GW-theories. We provide details for the case $E = A^6$.

First, we need to determine the cohomology ring $A^*(E)$. Pull-backs to $\mathbf{P}^9 \times G$ of the Chern classes $p = c_1(\mathcal{O}_P(1))$ and $\gamma_i = c_i(Q)$, $i = 1, 2, 3$, generate the \mathbf{Q} -algebra $A^*(\mathbf{P}^9 \times G)$, and induce generators of $A^*(E)$.

Lemma 2.

- i) *The \mathbf{Q} -algebra $A^*(E)$ is generated by p and γ_2 with top degree monomial values as follows:*

$$p^6 = 14 \quad p^4 \gamma_2 = 28 \quad p^2 \gamma_2^2 = 59 \quad \gamma_2^3 = 117$$

In particular the Betti numbers of $A^(E)$ are $[1, 1, 2, 2, 2, 1, 1]$.*

- ii) *We have the relation $\gamma_1 = 2p$.*

Proof. Computed using Schubert.⁴

□

Our choice of basis $\{\Delta_i\}$ for $A^*(E)$ is the following:

$$\{1, p, p^2, \gamma_2, p^3, p\gamma_2, p^4, p^2\gamma_2, p^5, p^6\}.$$

Rather than to work directly with the solutions (10) and (11) we prefer to work with their governing differential equations. The quantum differential equation (12) is determined by the two-point numbers $\langle \Delta_i, \Delta_j \rangle_d^E$ for all d in $A_1(E, \mathbf{Z}) \simeq \mathbf{Z}$. A simple dimension count shows that these are 0 unless

$$(13) \quad \text{codim}(\Delta_i) + \text{codim}(\Delta_j) = 5 + 3d.$$

Lemma 3. *Values of $d \geq 1$ GW-invariants satisfying (13) are as follows:*

$$\begin{aligned} \langle p^2, p^6 \rangle_1^E &= 238 & \langle p\gamma_2, p^5 \rangle_1^E &= 2044 & \langle p^2\gamma_2, p^2\gamma_2 \rangle_1^E &= 6617 \\ \langle \gamma_2, p^6 \rangle_1^E &= 504 & \langle p^4, p^4 \rangle_1^E &= 1568 & & \\ \langle p^3, p^5 \rangle_1^E &= 980 & \langle p^4, p^2\gamma_2 \rangle_1^E &= 3220 & \langle p^5, p^6 \rangle_2^E &= 9800 \end{aligned}$$

Proof. It follows from Lemma 2 that the map (4) identifies d with the curve class $(d, 2d)$ in $A_1(\mathbf{P}^9 \times G, \mathbf{Z})$, so the GW-invariants are integrals over $(\mathbf{P}^9 \times G)_{(d, 2d)}$.

Localization. Consider the standard action of $T = (\mathbf{C}^*)^{10} \times (\mathbf{C}^*)^7$ on $\mathbf{P}^9 \times G$. Since the integrands are polynomials in Chern classes of equivariant vector bundles with respect to the induced T -action on $(\mathbf{P}^9 \times G)_{(d, 2d)}$, they

⁴A MAPLE package for enumerative geometry written by S. Katz and S.-A. Strømme. Software and documentation available at <http://www.math.okstate.edu/~katz/schubert.html>.

may be evaluated using Bott's residue formula (see [10, 20]). As there are only finitely many fixed points and curves in $\mathbf{P}^9 \times G$, the formulae involved are similar to the ones found in [20]. Details are left to the reader. The two-point integrals with $d = 1$ were evaluated in this manner.

Reconstruction. The number $\langle p^5, p^6 \rangle_2^E$ is however more easily obtained from $d = 1$ numbers using the following WDVV-relations:

$$(14) \quad \left(\begin{array}{c} p \\ p^2 \gamma_2 \end{array} \right) \text{---} \left(\begin{array}{c} p^5 \\ \gamma_2 \end{array} \right)_2, \left(\begin{array}{c} p \\ p^4 \gamma_2 \end{array} \right) \text{---} \left(\begin{array}{c} p^5 \\ \gamma_2 \end{array} \right)_2, \left(\begin{array}{c} p \\ p^3 \gamma_2 \end{array} \right) \text{---} \left(\begin{array}{c} p^6 \\ \gamma_2 \end{array} \right)_2, \left(\begin{array}{c} p \\ p \gamma_2 \end{array} \right) \text{---} \left(\begin{array}{c} p^6 \\ \gamma_2 \end{array} \right)_2.$$

In fact, a further analysis⁵ shows that the 3-point $d = 1$ numbers appearing in the equations (14) are in turn determined by the 2-point $d = 1$ numbers above. \square

Remark. Employing a description in [9] of the class in $\text{Grass}_2(\wedge^2 V)$ of lines on M , the $d = 1$ GW-invariants which only involve powers of p can be computed without using Bott's formula.

Consider the differential equation (12), with invariants as in Lemma 3 and $\hbar = 1$. Denote $q = q_1$ and $D = q \frac{d}{dq}$. By reduction we find the (order 10, degree 5)-differential equation $P(D) = \sum_d q^d P_d(D) = 0$, with P_d as below, for $J_E(\hbar = 1)$.

$$\begin{aligned} P_0 &= 3D^7(D-1)^3, \\ P_1 &= D^3(194D^7 - 776D^6 + 1072D^5 - 1405D^4 - 1716D^3 - 1272D^2 - 414D - 51), \\ P_2 &= 343D^{10} - 1715D^9 + 3185D^8 - 58593D^7 - 55484D^6 - 460D^5 + 10697D^4 + \\ &\quad 1850D^3 - 896D^2 - 480D - 96, \\ P_3 &= -99127D^7 + 22736D^5 - 11772D^4 - 34797D^3 - 31654D^2 - 13495D - 2175, \\ P_4 &= -19551D^4 - 39102D^3 - 31360D^2 - 11524D - 1430, \\ P_5 &= 343(D+1). \end{aligned}$$

Let $H = 3\mathcal{O}_P(1)$ and assume $\hbar = 1$. The adjunct I_E^H is obtained by correcting the q^d -coefficients of J_E with the class $H_d = \prod_{m=1}^d (p+m)^3$. A reformulation of this transformation on the corresponding differential equations takes the same form⁶. That is, the differential operator

$$(15) \quad \sum_{d=0}^5 q^d P_d \prod_{m=1}^d (D+m)^3$$

annihilates I_E^H . Recall that the commutation rule is $Dq - qD = q$. If we factor out the “trivial” term $D^3(D-1)^3$ from the left of (15) and re-organize the terms we recover the Picard-Fuchs operator (1).

Proof of Theorem 1. From (9) it follows that $\langle p, p, p \rangle_d^{M^3} = \langle p, p, p \rangle_d^{A^3}$. Using (12) it is straightforward to check that $D^2 \frac{1}{K} D^2$ is the differential

⁵For instance using Farsta, a computer program written by A. Kresch, available at <http://www.math.upenn.edu/~kresch/computing/farsta.html>.

⁶This is a general principle when $\text{rank } A^1(E) = 1$. It is easily proved using recursion formulas for solutions of differential equations [3, 26].

operator governing J_{A^3} (see for instance [26]). The rest follows from Theorem 2. \square

The first five curve numbers are:

$$n_1 = 588 \quad n_2 = 12103 \quad n_3 = 583884 \quad n_4 = 41359136 \quad n_5 = 3609394096$$

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